ON THE ANALOGY BETWEEN PULSATING AND VIBRATING FLOWS

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The paper presents theoretical basis for the analogy between the pulsating and vibrating flows enabling a common rheodynamic theory to be formulated for both types of the flow.

Numerous recent papers have been dealing with the study of the effect of oscillating shear on time-averaged characteristics of axial flows in closed[1-7] and open[8-10] ducts. Shearing oscillations may be generated either by pulsations of the pressure gradient (pulsating flows) or vibrations of the walls (vibrating flows). Pulsating and vibrating flows have been studied thus far, with the only exception[7], separately without attempt for confrontation. It will be shown that in case of an incompressible fluid an analogy exists between both types of these flows enabling formulation of a common dynamic theory.

Noninertial Frame of Reference

In an inertial frame of reference for the steady axial flows in question the cartesian coordinates (x, y, z) are taken so as to have the outline of the duct in an arbitrary plane $z =$ const. represented by a curve $\Gamma_w \in (x, y)$. For simplicity we shall confine ourselves to axial flows with such a degree of symmetry that we may write

$$v_x = v_y = 0\,, \quad v_z = v(t;\, x, y)\,, \tag{1a,b,c}$$

$$-\varrho^{-1}\,\partial_z p + g_z = G(t)\,. \tag{2}$$

Under these simplifying assumptions the equation of motion may be reduced to

$$\varrho(\partial_t v - G(t)) = (\nabla\,.\,\tau)_z = \partial_x \tau_{xz} + \partial_y \tau_{yz}\,. \tag{3}$$

The deformation stress τ is given by the appropriate constitutive functional of the deformation stress history

$$\tau = \mathbf{T}[\mathbf{D}(s)]^t_{-\infty}\,, \tag{4}$$

where

$$\mathbf{D} = (\mathbf{e}_x\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_x)\,\partial_x v + (\mathbf{e}_y\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_y)\,\partial_y v\,. \qquad (5)$$

Apart from the initial and symmetry conditions, the mathematical formulation of the problem comprizes inhomogeneous boundary conditions on the wall of the duct

$$v_z = u(t)\,; \quad (x, y) \in \Gamma_w\,, \qquad (6)$$

where $u(t)$ stands for the axial velocity of the duct walls with respect to the inertial frame of reference.

Let us consider now the transformation properties of the set (3)–(6) undergoing the following transformation

$$z \mapsto z' = z + L(t) \qquad (7a)$$

$$v \mapsto v' = v + \dot{L}(t) \qquad (7b)$$

$$G(t) \mapsto G'(t) = G(t) + \ddot{L}(t)\,. \qquad (7c)$$

$L(t)$ is a parameter of this transformation which represents only a special case of the Newtonian–Galilei transformation[11,12]. The relationships between the parameters t, x, y, v_x, v_y in the original and the transformed system are identities, *i.e.* $t = t'$, *etc.* Clearly, the objective quantities $\mathbf{D}$ and τ are invariant with respect to the transformations (7a,b,c). As a special case, $\mathbf{D}' = \mathbf{D}$ in view of $\partial_x v = \partial_x v'$, $\partial_y v = \partial_y v'$. With respect to these transformations, the following equation of motion appears also invariant:

$$\varrho(\partial_{t'} v' - G'(t')) = (\nabla' \cdot \mathbf{T}[\mathbf{D}'(s)]_{-\infty}^{t'})_z\,. \qquad (9)$$

The transformations (7a,b,c) shall influence only the structure of the boundary conditions on the wall of the duct

$$v' = u' = u(t) + \dot{L}(t)\,; \quad (x, y) \in \Gamma_w\,. \qquad (10)$$

The described transformation may be comprehended physically as a transition from the inertial frame of reference to a new one, moving with respect to the original frame in the axial direction at the velocity $\dot{L}(t)$, *i.e.* with an acceleration $\ddot{L}(t)$. This acceleration shall become manifest through the fictious body forces in accord with Eq. (7c). The flow situations satisfying the following identities, $G'(t) =$ idem, $u'(t) =$ = idem, under suitable transformations $L(t)$ represent the class of equivalent motions.

Periodic Axial Flows

Consider the case where in an inertial frame of reference there acts, aside from the constant mass acceleration $G_0 =$ const., an oscillatory component $G_1 = G_1(t)$ and the axial velocity of the walls of the duct may be also split into a steady and a periodic component of equal period θ:

$$G(t) = G_0 + G_1(t), \quad u = u_0(t) + u_1(t) \tag{10a,b}$$

$$G_1(t + \theta) = G_1(t), \quad u_1(t + \theta) = u_1(t) \tag{11a,b}$$

$$\int_0^\theta G_1 \, dt = 0, \quad \int_0^\theta u_1 \, dt = 0. \tag{12a,b}$$

In order that we may study all periodic axial flows generally, we shall select that canonic frame of reference for which we may write

$$G' = \text{const.} \tag{13a}$$

$$u'_0 = 0. \tag{13b}$$

In this frame of reference, all acting external effects formally decompose in such a manner that the liquid is under the effect of only time-independent fictious body force or pressure gradient and the walls of the duct oscillate with respect to the frame of reference about a mean position. Corresponding canonic transformation $L(t)$ may be always found: From Eq. (*7*) through (*13*) we have

$$\dot{L}(t) = -u_0 + \theta^{-1}\left(\int_0^\theta G_1(s)\, s \, ds + \int_0^t G_1(s)\, \theta \, ds\right), \tag{14}$$

$$G' = G'_o = G_0 \tag{15a}$$

$$u' = u'_1 = u_1(t) + \theta^{-1}\left(\int_0^\theta G_1(s)\, s\, ds + \int_0^t G_1(s)\, \theta \, ds\right). \tag{15b}$$

The relationships (*14*) and (*15*) represent basic results we have been aiming for. If $G'_0(t)$ = idem, $u'(t)$ = idem, we have for a given material and geometry of the duct also $v'_z = v'(t, x, y)$ = idem.

Example. Consider the laminar flow of a viscoelastic fluid in a horizontal ($g_z = 0$) flat duct with walls at $x = \pm h$ under two following situations:

Case A. The pressure gradient is steady and the walls of the duct oscillate harmonically in the axial direction:

$$-\varrho^{-1}\, \partial_z p = G_0$$

$$v = a\omega \cos(\omega t) \quad \text{for} \quad x = \pm h \tag{16a,b}$$

Case B. The walls of the duct are immobile and the pressure gradient oscillates harmonically about the mean G_0:

$$v = 0 \quad \text{for} \quad x = \pm h$$
$$-\varrho^{-1} \, \partial_z p = G_0 + G_c \sin(\omega t) . \qquad (17a,b)$$

The velocity field in both cases considered possesses a high degree of symmetry: $v_x = 0$, $v_y = 0$, $v_z = v(x)$. The non-zero component of the rate-of-strain tensor in the system (x, y, z) is clearly $\gamma(t, x) = \partial_x v$:

$$\mathbf{D}(t, x) = \partial_x v(\mathbf{e}_x \mathbf{e}_z + \mathbf{e}_z \mathbf{e}_x) . \qquad (18)$$

Assuming that the anisotropic normal stresses are balanced by the appropriate reactions of the walls, the equation of motion reduces to the scalar form *(3)* with $\tau_{yz} = 0$.

For the description of the dynamics of the flow we shall introduce a new (primed) frame of reference differing from the original one only in the definition of z'.

$$z' = \begin{cases} z \,; & \text{case } A & (19a) \\ z + P_c \omega^{-2} \sin(\omega t) \,; & \text{case } B . & (19b) \end{cases}$$

In these new coordinates the equation of motion assumes a new form (according to the Galilei principle of relativity) where instead of $g_z = 0$ we have

$$g'_z = \begin{cases} 0 \,; & \text{case } A \\ 0 - P_c \sin(\omega t) \,; & \text{case } B . \end{cases} \qquad (20)$$

Putting

$$a' = \begin{cases} a \,; & \text{case } A & (21a) \\ P_c \omega^{-2} \,; & \text{case } B & (21b) \end{cases}$$

the model of the flow in the coordinates (x', y', z', t')) takes a common form for both cases considered

$$\varrho(\partial'_t v' - G_0) = \partial'_x \tau'_{zx} \qquad (22a)$$

$$v' = a' \omega \cos(\omega t) \,, \quad x' = \pm h \qquad (22b)$$

$$v'(x', t' + 2\pi/\omega) = v'(x', t') . \qquad (22c)$$

LIST OF SYMBOLS

$\mathbf{D}$	rate-of-strain tensor
g_z	axial acceleration in an inertial frame of reference
$G(t)$	potential of effective driving forces for the axial flow, Eq. *(2)*
$L(t)$	mutual axial displacement between the inertial and a new frame of reference
p	isotropic pressure
s, t	time
$\mathbf{T}$	rheological constitutive operator
u	axial velocity of fluid on wall, velocity of wall

v	axial velocity
v_x, v_y, v_z	cartesian velocity components
x, y, z	cartesian coordinates
$\partial_x, \partial_y, \partial_t$	operators of appropriate partial derivatives

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